Effect of Resources on the Efficiency-Fairness Tradeoff for Allocation Problems

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Efficiency and fairness often conflict in resource allocation. In this work, we study how the technical decisions made about operationalizing efficiency and fairness affect the perceptions of the tradeoffs between them. We characterize the effect of resources on group utilities and the evaluation measures through the framework of homogeneous functions. Then, we explore the tradeoff together with three decision choices — allocating resources based on needs vs. outcomes, measuring social welfare as maximizing vs. minimizing objectives, and expressing fairness in terms of absolute vs. relative measures. Each choice presents us with a rich, complex setting and different desirable solutions. We illustrate our findings through stylized examples and then a more realistic example based on restaurant inspections.

CCS Concepts: • **Applied computing → Multi-criterion optimization and decision-making**; Economics; • **General and reference** → *Evaluation.*

Additional Key Words and Phrases: Fairness, Tradeoffs, Resource Allocation

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1 Introduction

Resource allocation with respect to efficiency and fairness is often studied under the assumption of a fixed pool of resources. However, organizations often need to make a case for the benefits of being supplied with additional resources (e.g., when applying for grants). Similarly, a government facing budget cuts will care about the implications of reducing the resources provided to various departments. In such situations, it is not immediately clear *how the change in resources impacts efficiency and fairness,* since the change can affect efficiency and fairness not only individually but also together. For example, imagine a local government applying for a federal grant to improve public transit — the funds could be used to improve service (efficiency), start new lines that improve access (fairness), or improve both service and access (efficiency and fairness). Our paper approaches this problem by studying the change in the tradeoff between efficiency and fairness with a change in resources.

One might imagine that the effects of a change in resources could be understood simply by using any standard approach to the trade-off between efficiency and fairness at the new level of resources. As one example, we could study what would happen if we always chose the most efficient way to deploy the resources and compare it to the fairest way as is done, e.g., in the literature on the Price of Fairness [4, 10] (see Appendix A). Perhaps surprisingly, we find that such comparisons depend on subtle details of the way fairness and efficiency are measured. For example, given the *exact same* scenario different measurements will conclude that fairness increases linearly, sublinearly, or not at all with efficiency as resources are added. To study this interaction between resources and

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objectives, we zoom into the specific context to understand (*i*) how the tradeoff changes with a change in the amount of resources, (*ii*) how different definitions of societal objectives — efficiency and fairness — change the observed effect of increased or decreased resource allocation.

On a technical level, we address (*i*) by explaining the effect of changing resources through the lens of homogeneity, a property of some functions that determines how they behave when scaling up their inputs. We observe that many common choices for utility functions and fairness measures are homogeneous and demonstrate how the degree of homogeneity affects the nature of tradeoffs. We use homogeneity to make several observations about tradeoff analysis. These observations address (*ii*) by highlighting how seemingly small technical details can lead to substantially different quantifications of tradeoffs and, thus, different decisions. In particular, we focus on the following three decisions:

- Should we measure utility in terms of the needs fulfilled or the outcomes created by fulfilling that need?
- Should we frame the optimization in terms of maximizing benefits or minimizing a cost function?
- Should we measure fairness violations in terms of the absolute or relative disparity in utilities?

All of these questions capture some aspect of the fact that efficiency and fairness are not dimensionless measures and none of these questions has an obvious right answer. Most combinations of answers seem reasonable for at least some plausible situation. But we expose how these technical decisions have important implications.

Need vs. outcomes captures the difference between analyzing the number of resources deployed to, e.g., each region of a city (adjusted based on population and other factors relevant to determining "need") and what positive benefits the resources produce. From a mathematical perspective, the simplest version of need is linear: each additional resource produces a constant benefit; measuring output is more complex but allows us to capture a form of diminishing returns: going from one street sweeper to two halves the time between cleanings, but going from two to three reduces it by a third. We show through a simple stylized example that the two formulations have very different implications for the fairness benefits of adding resources. In the former case adding resources will tend to exacerbate unfairness, as the efficient solution will concentrate them in the neediest region, while in the latter case the diminishing marginal returns of resources tend to lead to improved fairness when adding resources. Our point is not that one of these two choices is "right" (that may depend on what better captures the preferences of stakeholders in the situation), but rather that this simple modeling choice can substantially affect the way adding resources affects the tradeoff between efficiency and fairness.

Similarly, we could equally well define a utility function to capture the benefits of allocating resources or a societal cost function to capture the cost of only having allocated a limited number. (In the street sweeping example, this cost function might capture the amount of time the streets remain uncleaned.) This question turns out to have a non-obvious linkage to the third. We argue that the combination of measuring the utility of resources in terms of the reduction in societal costs and using relative disparity to measure fairness often leads to unreasonable conclusions about tradeoffs. In particular, it suggests that even with ample resources, so that there is little societal cost to any group, it is still possible for the efficient allocation to be extremely unfair. Returning to our earlier example, consider a city with two regions, region A and region B, receives a huge grant to hire more street sweepers such that it can reduce the overall street sweeping times. In region A, the sweeping time reduces from 2 days to 6 hours, and in region B, from 1 day to 3 hours. Although more resources are allocated to region A, we still reduce the disparity when measured in absolute terms and the overall time for which streets stay unclean. In relative terms, such as the ratio of sweeping times between the regions, the disparity indicates no improvement. We argue this is unreasonable; at some point, the gap will be close enough that most people will agree that the outcome seems reasonably fair in practice, even if some difference remains.[1]

[1]If six versus three hours still seems substantially unfair to you, what about six versus three minutes? Seconds? At some point, the claim that fairness has not improved becomes untenable.

In contrast, when a simple transformation of the efficiency axis reframes the optimization in terms of benefit, the *exact same scenario* would be evaluated as near-perfect fairness at even low resources. That is, if there are 30 days in a month then at the initial level of resources region A has 28 "clean days" while region B has 29, a ratio much closer to perfect fairness. This issue disappears when using absolute disparity to quantify fairness. However, absolute disparity has its own downsides, such as being less interpretable. Such caveats complicate making a recommendation in isolation and desire a more nuanced consideration. We propose that the choice of these different perspectives with changing resources can be understood by examining the homogeneity (or lack thereof) in the relevant definitions. In particular, homogeneity highlights the avoidable pitfalls of combining relative disparity and societal costs when resources can change.

Having illustrated these factors using a stylized model, we provide a more general theoretical characterization and then show their relevance in a more realistic decision-making scenario with discrete, indivisible resources by examining their effects on the optimization of restaurant inspection schedules based on data from Chicago [32, 33]. We observe behavior broadly consistent with our homogeneity-based predictions.

In summary, our main contributions are:

- Proposing using the homogeneity of many common utility functions and objectives as a tool for understanding the effects of adding resources on the efficiency-fairness tradeoff. (§3 & 4)
- Demonstrating through stylized examples and a general characterization that three technical choices can have a substantial effect on the shape of these tradeoffs and that homogeneity can help explain this. (§5 & 6)
- Confirming that these insights are relevant in a scenario based on a real dataset that does not perfectly satisfy our stylized assumptions. (§7)

2 Related Work

2.1 Resource Allocation

Resource allocation has been studied in several forms and disciplines. Fair division literature has offered some notions of fairness that have been implemented in several applications. For example, Ghodsi et al. [16] uses dominant resource fairness for scheduling computing tasks. For heterogeneous resources, researchers have used envy-based fairness notions [35] and extended them to group fairness [1]. Gölz et al. [20] have discussed the compatibility of a popular fairness definition in machine learning, namely equalized odds, with the axioms of classic fair division. Another prominent area for resource allocation is policy-making [3, 31]. Several works have proposed simulation studies to determine fair resource allocation for government agencies [29, 31, 33], long-term fairness goals [9], COVID-19 testing [3], kidney exchange [28], and under censored feedback [13]. Researchers have also studied counterfactual utilities for resource allocation [5, 25]. Our work is inspired by these applications and studies the relationship between group utilities and scalable resources that determine the efficiency and fairness of a resource allocation solution. A more recent paper by Goethals et al. [19] formulates positive predictions as fixed resources and introduces the cost of fairness as a function of base rates. Although we support their findings of considering resource levels for fairness tradeoffs, our paper goes beyond predictions and studies the efficiency and fairness induced by group utilities. In the context of resource allocation, Farahi et al. [14] have suggested a group fairness definition based on a binary beneficiary state of agents and propose an optimization model in dynamical system. We complement their findings by showing the cases where an efficient allocation converges to a group fair solution and supplement the analysis through additional measures of unfairness and outcomes.

2.2 Tradeoffs

The tradeoff between efficiency and fairness is intrinsic to decision-making with multiple objectives. Prior works have analyzed the tradeoffs between multiple fairness definitions [15, 22, 24] and with performance and fairness measures [2, 6, 7, 21, 26, 30, 33]. Relevant work by Mashiat et al. [27] presents resource allocation settings closest to ours with housing services that have non-uniform utility and capacities. They propose four fairness definitions that can be written as differences and ratios. Their findings state that the same allocation can portray different groups as favored based on the chosen definition. Unlike their definition of tradeoff, which only considers fairness, we study the tradeoff between efficiency and fairness and the effect of resources on it.

3 Utility as Homogeneous Functions

Suppose T different types of resources have to be allocated to G groups for $g \in \{1, \ldots, G\}$. The number of resources of each type is n_t for $t \in \{1, \ldots, T\}$. Although each resource type is different, the resources within each type are assumed to be uniform and interchangeable. Mashiat et al. [27]'s work on housing services illustrates one such setting where multiple resources capture different types of services. However, our formulation also works for cases with a single type of resource, for example, the number of public libraries in a neighborhood. Each group g receives some amount of each resource type and derives a benefit from this, which we call *group utility* denoted as u_g. We define *total utility*, or *social welfare* as an aggregate of the group utilities, which is a measure of total benefit drawn by all the groups from the allocated resources. An interesting property of the group utilities we explore in this section is homogeneity. When the group utilities are homogeneous functions of the resources allocated to them, it implies that if the allocated resources to a group are scaled by a factor of s, the resulting group utility is scaled by the same factor raised to some power k.

Definition 1 (Homogeneous Function). *A function $f : \mathbb{R}^T_{>0} \mapsto \mathbb{R}$ is (positive) homogeneous of degree k if $f(sX) = s^k f(X)$, where s is a (positive) scalar and X is a T-dimensional vector of (positive) real numbers.*

Homogeneous utility functions allow us to model varied effects of the change in resources using group-specific functions. Even with a single type of resource and allocation r_g to a group, we can capture a wide variety of forms for $u_g(r_g)$ including linear, monomials such as r_g^2, and even inverse relationships such as $\frac{1}{r_g}$ if it is an undesirable thing such as polluting industrial activity. In a resource allocation setting, our objectives would combine (in the case of efficiency) or compare (in the case of fairness) the benefit resource units $\mathbf{r} \in \mathbb{R}^T_{\geq 0}$ bring to the groups. If we have an objective function f measuring fairness or efficiency that is also homogeneous, the resulting combination is homogeneous as well:

Proposition 1. *If the group utilities are homogeneous functions of degree k_1, and the objective function is defined as a homogeneous function of degree k_2 over the group utilities, then the resulting combination is a homogeneous function of degree $k_1 k_2$.*

Using Proposition 1, we present two applications where the choice of the objective function can lead to different outcomes based on the change in group resources. First, consider a linear function of the group utilities as an objective. This is common when considering efficiency measures like social welfare and fairness measures that consider the difference in group utilities.[2] A linear function is homogeneous of degree 1. Thus, the effect of change in resources on linear functions is to scale the objective accordingly.

[2]Defining fairness in terms of group utilities is a way to extend standard group fairness notions beyond classification settings [5], and has been used in prior work on resource allocation [10, 18, 33]

Corollary 1. *If the group utilities are homogeneous functions of degree k, then a linear function of the group utilities is also a homogeneous function of degree k;*

$$f(u_1(s\mathbf{r}_1), \ldots, u_G(s\mathbf{r}_G)) = s^k f(u_1(\mathbf{r}_1), \ldots, u_G(\mathbf{r}_G)) \ .$$

Second, we explore another natural choice for objective functions, a ratio of the group utilities. Ratios are particularly amenable when quantifying comparisons as they are relative – compared to a known quantity (denominator), independent of units, and could be expressed in fixed intervals, making them easy to understand. Since ratios are defined over two quantities, for this example, let us assume $G = 2$, but our results extend to any number of G via simple transformations, for example, comparing extrema over a set.[3]

Corollary 2. *If the group utilities are homogeneous functions of degree k, then their ratio is a homogeneous function of degree zero. Thus,*

$$\frac{u_1(s\mathbf{r}_1)}{u_2(s\mathbf{r}_2)} = \frac{u_1(\mathbf{r}_1)}{u_2(\mathbf{r}_2)} \ .$$

An important implication of this corollary is that $f(\cdot)$, when calculated as a ratio of the group utilities, the result is independent of any changes in the total resources $\mathbf{r}$. This result highlights an important caveat of using ratio as an objective function, and we further examine the implications of this behavior in the later sections.

Of course, these results only apply when the resources allocated to all groups are scaled up by the same constant. It is not immediately obvious that we should do so; perhaps it would be better to allocate the additional resources disproportionately to one group. However, for homogeneous functions, scaling up evenly is optimal.

Proposition 2. *Let $\mathbf{r}^*$ be the optimal allocation of some quantity of resources given group utilities which are homogeneous of degree k_1 and an objective f which is homogeneous of degree k_2. Then given s times as many resources the optimal allocation is $s\mathbf{r}^*$.*

Proof. Any other allocation can be written in the form $s\mathbf{r}$ where $\mathbf{r}^*$ and $\mathbf{r}$ allocate the same total amount of each resource. WLOG assume we wish to maximize f. Then we have:

$$\begin{aligned} f(u_1(s\mathbf{r}^*{}_1), \ldots, u_G(s\mathbf{r}^*{}_G)) = s^{k_1 k_2} f(u_1(\mathbf{r}^*{}_1), \ldots, u_G(\mathbf{r}^*{}_G)) &\geq s^{k_1 k_2} f(u_1(\mathbf{r}_1), \ldots, u_G(\mathbf{r}_G)) \\ &= f(u_1(s\mathbf{r}_1), \ldots, u_G(s\mathbf{r}_G)), \end{aligned}$$

where the two equalities follow by Proposition 1 and the inequality follows because $\mathbf{r}^*$ maximizes f for the given quantity. □

Omitted proofs are presented in Appendix C.

4 Modeling Fairness with Expanding Resources

The total amount of resources is one of the largest constraints when designing the algorithm for a socio-technical problem. This is often presented as an immutable fact and, therefore, often overlooked when considering the desirable efficiency and fairness properties of an algorithm. However, we argue that when resource inevitably change (from funding grants or budget cuts), the change in fairness and its tradeoff with efficiency depends on the modeling choices.

Let us revisit the example of a local government applying for a federal grant (see §1) that makes more resources available. Naively applied, without changing any algorithm parameterization, more resources may increase efficiency but lower fairness or vice versa. Or more resources can simultaneously increase both efficiency and fairness. More broadly, we could retune our algorithm and the exact set of choices available to us would vary based on the upper and lower bounds of the solutions to the multi-objective optimization.

[3]Demographic Parity Ratio (DPR) for more than two groups is often defined as the worst case, min-max ratio [17, 36]. See also the notion of relative fairness in Section 6.

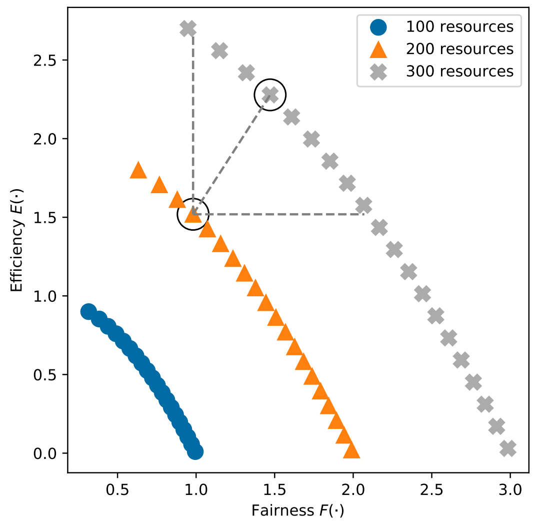


Fig. 1. Hypothetical Efficiency-Fairness tradeoffs at varying amounts of resources. The circles joined by the dashed lines show one possible increase in fairness and efficiency obtained by increasing the resources. The other dashed lines show alternate possibilities.

We show a hypothetical example of the transformation to the relationship between efficiency and fairness based on the resources available in Fig. 1. On the X-axis, we show a hypothetical homogeneous fairness function $F(\cdot)$, and on the Y-axis, a homogeneous efficiency function $E(\cdot)$. Each tradeoff curve shows the scaling of the efficiency and the fairness functions for $100, 200, 300$ resources. Each curve presents a different Pareto front offered by scaling the resources and each individual marker shows one possible allocation along with its efficiency and fairness. Thus, this figure captures how the tradeoff curve changes with resources, in this case keeping its shape (as in Corollary 1).

Let us say our current amount of resources is 200 (orange triangles), and our current operating point is illustrated by the circle on the tradeoff curve in Fig. 1. The dashed lines between the operating point associated with 200 resources and the one on the 300 curve (gray crosses) show a possible improvement along both axes – efficiency (Y-axis) and fairness (X-axis), both individually and together, from a grant that added 100 resources. If we use the increase in resources purely to improve fairness, the resulting improvement is essentially the same as keeping the resources the same and moving fully to the fair end of the Pareto frontier. Similarly, the efficiency achieved by the increased resources (vertical dashed line) is beyond what is possible with 200 resources. Faced with this scenario, we would argue that the most important decision is likely not the tradeoff between efficiency and fairness but the tradeoffs involved in getting more resources. Intuitively, it is unsurprising that adding resources can benefit fairness. What is more subtle (and our homogeneity approach addresses) is the need to understand how modeling formulations mediate the benefits of resource changes. For example, using this same example (Fig. 1) in Appendix A, we show how the Price of Fairness can lead to incomplete conclusions about the tradeoffs and how homogeneity can explain this.

Of course, our homogeneity-based framework only provides insight to the extent that the group utilities and metrics of interest are homogeneous. In Sec. 3 we discussed natural stylized forms of group utility that satisfy this including linear, monomials, and even, inverse relationships. We will see in Sec. 7.1 that the more realistic examples we consider have group utilities that approximately behave like these stylized examples. Prominent related work that discuss resource allocation and its benefit to groups could also be explained via our homogeneity-based framework. For example, Gölz et al. [20] examine resource allocation when resources are allocated within groups and only some individuals in each group benefit from the resource. They assume a calibrated classifier can predict how likely each individual is to benefit, and consider different within-group allocation strategies to achieve fairness. Similarly, Mashiat et al. [27] consider a setting with multiple of resources and different individuals receive different benefits for each resource. In these settings, the overall utility of each group depends on how resources are allocated internally within each group. Both consider two different styles of algorithm yielding different forms of utility function. One is to give out the resources uniformly at random. Since all individuals are equally likely to get resources, group utility grows linearly with resources (at least until there are so many resources that every individual has one), which is homogeneous of degree 1. They also consider both constrained and unconstrained forms of allocation that try and maximize the benefit to each group by allocating resources to individuals that are most likely to benefit from them. This means that group utilities show diminishing returns as later resources get allocated to individuals progressively less likely to benefit from them. This type of behavior can be captured with a monomial with an exponent of, say, 1/2 or 3/4. The coefficient can then capture differences between the groups in the base rate of individuals who would benefit.

In terms of objectives, our work, as well as those of Gölz et al. [20] and Mashiat et al. [27] focus on objectives inspired by the literature on group fairness. For instance, (utilitarian) social welfare is homogeneous of degree 1, while two natural metrics capturing the extent to which a group fairness definition, such as demographic parity or equal opportunity, is violated (absolute fairness and relative fairness) are homogeneous of degree 1 and 0 respectively. These concepts are defined formally in Sec. 5 and 6 in Eqs. (2), (3), and (5) respectively. For additional discussion, including a broader range of group utilities and metrics and more specifics on how we interpret group fairness notions in this context, see Appendix B.

Even when homogeneity does not hold for a particular group utility or metric, there may be some homogeneous substructure that can be exploited. For example, in general, polynomials are not homogeneous but their leading monomial which dominates their asymptotic behavior is. In a particular case of this observation, if the lack homogeneity is due to a constant, this can be mitigated by analyzing differences (which cancel out the constant) or analyzing the behavior of the homogeneous substructure without the constant. See Theorems 1 and 2 respectively for examples of these strategies.

5 Utility — Need vs. Outcome

Imagine a scenario in which groups have different utilities and are known to us. We want to explore how the choice of measuring the efficiency based on the *needs* of those groups compares to the *outcomes* those groups desire. Such alternative ways to calculate efficiency can present decision-makers with different magnitudes of feasible, competing measures.

Consider two groups, $G = 2$, among which we need to divide the n resources of a common type $|T| = 1$, and values of n range from 2 to 10. Suppose we know the two groups derive different utilities, denoted as $\mathbf{u}$ of size G, gained from receiving some amount of resource, such that $\mathbf{r}$ represents the allocation of resources to said groups. We define their *need*-based utilities as:

$$\mathbf{u}_1 = 2\mathbf{r}_1 , \quad \mathbf{u}_2 = \mathbf{r}_2 . \tag{1}$$

When $\mathbf{r}_1 = \mathbf{r}_2$, group 1 derives twice as much utility from receiving resources as group 2. This assumption might reflect, for example, greater importance of the resource to group 1 or a greater capacity to use it effectively. Let us consider two methods of resource allocation among the two groups. An *efficient allocation* would assign all the resources to group 1, given its higher utility. On the other hand, a *fair allocation* would split the resources among the groups so that each group gets equal utility from their allocation. Note this is not the same as equal allocation of resources, as group 1 would get twice as much utility from the same amount of resources. Our goal is to measure the efficiency and fairness of the allocations. We compute the *efficiency* for the allocations by averaging

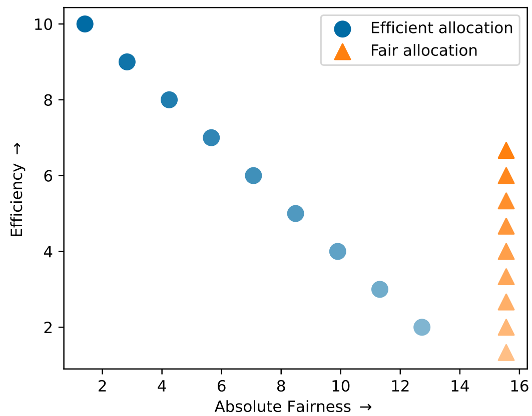


Fig. 2. The efficient allocation (blue circles) and the fair allocation (orange triangles) show the ends of the Pareto front representing an increasing number of resources. Efficiency and fairness are computed using *need*-based group utilities. Resources increase from transparent to opaque markers.

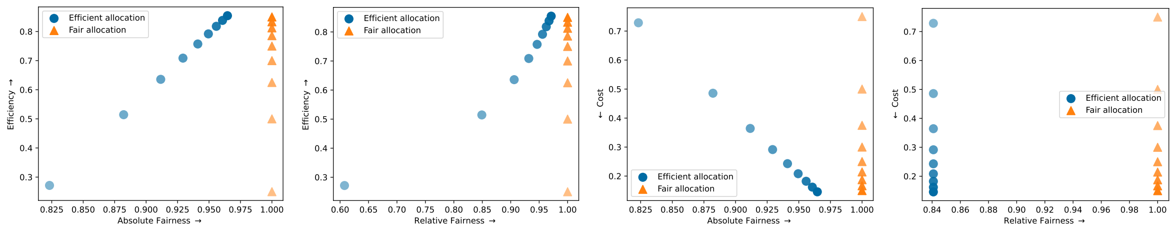
the sum of utilities over the groups. We can calculate the *unfairness* by using the L^2-norm over the difference between each group's utility and efficiency of the allocation. As is often done, we transform the unfairness into a maximizing fairness function by subtracting the unfairness from a constant, C, so that the fairness values are non-negative. For our example, we chose C based on the highest value of unfairness achieved across levels of resources, but C could be any arbitrarily high value to ensure boundedness. We call this difference-based measure of unfairness *absolute fairness*. The general version of efficiency and absolute fairness functions are given as follows:

$$E(\mathbf{r}) = \frac{1}{G} \sum_{i=1}^{G} \mathbf{u}_i \ , \tag{2}$$

$$F_{\text{abs}}(\mathbf{r}, E(\mathbf{r})) = C - \sqrt{\sum_{i=1}^{G} (\mathbf{u}_i - E(\mathbf{r}))^2} \ . \tag{3}$$

Absolute fairness is a widely adopted definition of (un)fairness for multi-group settings [11, 12], although the choice of the norm may vary; the version we use essentially considers the standard deviation of the utilities. We show the efficiency-fairness tradeoff curves in Fig. 2, where each pair of circles and triangles shows the ends of a Pareto front corresponding to one choice of n. The lowest blue and orange marks correspond to 2, while the highest corresponds to 10. We note that the fairness (X-axis) for the fair allocation (orange triangles) stays the same since regardless of the number of resources, we can choose a perfectly fair solution. However, the increase in resources produces an increase in the efficiency of this fair allocation. In contrast, the efficient allocation starts with lower absolute fairness, and as the resources are added, the fairness decreases as one group's utility dominates the allocation. We can characterize the effect of resources for efficient allocation as a property of homogeneity (see §3). Since efficiency, fairness[4], and both utilities are all homogeneous of degree 1, we observe both axes changing by a constant factor. As a result, the gap between the allocations with maximum fairness and maximum efficiency becomes wider as the resources increase. This example highlights the caveats

[4]It is only homogeneous without the constant C. As we are interested in the difference between points, the C naturally cancels.



(a) Absolute Fairness (b) Relative Fairness (c) Absolute Fairness for Costs (d) Relative Fairness for Costs

Fig. 3. The Pareto front with an increasing number of resources when efficiency (Y-axis) and fairness (X-axis) are computed using *outcome*-based group utilities.

of adding resources when they have a linear relationship with the group utilities and could be thought of as *need*-based allocation. A real-world example of such a scenario would be allocating the number of police officers to neighborhoods by the crime-per-capita metric. If a decision-maker is not deliberate about how the resources are assigned in this setting, the results might be much worse along one axis than the other.

Now let us consider another form of a utility function, *outcome*-based. Imagine each group has a fixed number of jobs that must be completed. In that case, adding an extra resource will speed up the completion time, increasing the group utility by the inverse factor. This yields a utility of the form:

$$\mathbf{u}_1 = 1 - \frac{1}{2\mathbf{r}_1}\,, \quad \mathbf{u}_2 = 1 - \frac{1}{\mathbf{r}_2}\,. \tag{4}$$

Analogous to *need*-based utility functions described in Eq. (1), group 1 benefits more from the same amount of resources than group 2. Using the same allocation mechanisms described above and the methods to calculate efficiency and absolute fairness, we plot the tradeoff curves for different resources in Fig. 3a. Unlike Fig. 2, the fairness (X-axis) of the efficient allocation increases as we add more resources, as increasing resources brings the utilities for both groups closer to 1. Because the efficient solution has now changed, the inverse relationship between the resources and group utilities causes the gap between the extreme points of the Pareto front—efficient allocation (blue circles) and fair allocation (orange triangles)—to shrink with the addition of resources. Since the (differences in) group utilities are homogeneous of degree -1, changes in measures of efficiency and fairness diminish with added resources. In this type of scenario, where there are diminishing returns to providing additional resources, with enough resources we would naturally converge towards a more efficient solution for a low cost of fairness. A real-world example includes buying more snowplowing machines for city neighborhoods. Adding more machines helps all the regions and the effect of *fair* or *efficient* allocation starts to vanish after a while.

Our point is not that either formulation of utility is better than the other. It is up to stakeholders whether the importance of, e.g., police is in providing a feeling of safety through their presence or in their effectiveness at preventing crime. Similarly, fairness in one case corresponds to no group feeling over- or under-policed relative to others, while the other corresponds to equalizing the crime experienced. All of these may be reasonable goals. Our point is that choosing one or the other (or even a combination) can have a substantial impact on the nature of efficiency-fairness-resource tradeoffs, and that homogeneity can help us understand the shape of impact we should expect.

6 Expression of Goals & Theoretical Analysis

In this section, we explore the example with the outcome-based utilities (Eq. (4)) to investigate how different expressions of efficiency and fairness can modify their purpose and choice for the best decision. We modify both, together and separately, the fairness measure from absolute to relative and switch the interpretation of efficiency from maximizing a utility function to minimizing a cost function.

In addition to *absolute* fairness (Eq. (3)), another expression for fairness is through ratios. We propose that with multiple groups *Relative* fairness could be defined as the geometric mean of the ratios of group utilities and allocation efficiency, given as:

$$F_{\text{rel}}(\mathbf{r}, E(\mathbf{r})) = \prod_{i=1}^{G} \left(\frac{\min(\mathbf{u}_i, E(\mathbf{r}))}{\max(\mathbf{u}_i, E(\mathbf{r}))} \right)^{\frac{1}{G}} \tag{5}$$

This relative expression of fairness is useful when the fairness goals are set based on the other group utilities and independent of the actual units. We show the efficiency-fairness tradeoff and resources in Fig. 3b. Similar to Fig. 3a, the Pareto front contracts as we increase the resources, and the efficient allocation moves closer to the fair allocation. However, the relative fairness measure shows that efficient allocation moves more towards fair allocation when adding resources at lower resource levels than the absolute fairness measure. In other words, the improvement in relative fairness appears to be sublinear compared to the linear growth with absolute fairness. This is despite the efficient and fair allocations being *exactly the same* in the two cases. Although both measures of fairness are not directly comparable, we can identify ways they lead to different conclusions. For example, the slope of the line between the efficient and fair solutions is steeper at low resource values and shallower at high resource values in Fig. 3b than Fig. 3a. This might lead to different decisions about how much utility to trade for fairness depending on the choice of absolute vs. relative.

Thus far, we have only looked at the efficiency measures that must be maximized for social good. Now, we explore the effect of a cost function that must be minimized. Continuing with our example of *outcome*-based utilities, we slightly change the group utilities keeping the privileged group as-is. They are given as follows:

$$\mathbf{c}_1 = \frac{1}{\mathbf{r}_1}\,, \quad \mathbf{c}_2 = \frac{1}{2\mathbf{r}_2}\,. \tag{6}$$

We compute the *societal cost* similar to the *efficiency* (Eq. (2)), but the utilities now aggregate to a quantity whose objective is transformed into a reduction in loss incurred to the society when a resource is unavailable. From an optimization perspective, this transformation is harmless: maximizing efficiency and minimizing cost are equivalent. Combining the homogeneity of cost (degree −1), societal cost (degree 1), absolute fairness (degree 1), and relative fairness (degree 0), we can now apply our theoretical results to these scenarios.

We contrast the cost vs. absolute fairness and relative fairness in Figs. 3c & 3d respectively. Consistent with the findings in Fig. 3a, Fig. 3c shows the shape of the Pareto front inverted across a line parallel to the X-axis, and it shrinks as the cost for the efficient and fair allocation reduces. Proposition 1 tells us that both axes are homogeneous of degree −1, which is why each doubling of the resources halves the gap to optimality on each. On the other hand, Fig. 3d illustrates an interesting result. When measuring relative fairness with costs, the fairness (X-axis) for efficient allocation (blue circles) and fair allocation (orange triangles) *do not change* with the increase in resources, and the tradeoff stays constant. This is because the ratio of group costs remains the same for both the efficient and fair allocations and is a direct application of Corollary 2. Again, the underlying efficient allocations in the two figures are *exactly the same.* However, conclusions drawn from the two figures would differ vastly and could lead to different choices. For example, by only looking at Fig. 3d, we could infer that the addition of resources has no effect on the fairness of the efficient allocation. Conversely, measuring efficiency with relative fairness (Fig. 3b) shows the increase in fairness comes at a low cost of efficiency with rising resources, which we view as a more reasonable reflection of the situation.

The key takeaway is that seemingly small technical decisions, such as the choice to measure fairness violations in relative vs. absolute terms, can have substantial effects on the nature of the efficiency-fairness tradeoff with the change in resources and that these effects can be understood through the lens of homogeneity. In Appendix D,

Metrics	Group Utility Functions			
	Need-based		Outcome-based	
	Utility	Cost	Utility	Cost
Goal	Maximize	Minimize	Maximize	Minimize
Form	$u(s\mathbf{r}) = s^k u(\mathbf{r})$	$c(s\mathbf{r}) = C - s^k u(\mathbf{r})$	$u(s\mathbf{r}) = C - s^{-k} c(\mathbf{r})$	$c(s\mathbf{r}) = s^{-k} c(\mathbf{r})$
Efficiency Gap	$\Theta(s^k)$	$\Theta(s^k)$	$\Theta(s^{-k})$	$\Theta(s^{-k})$
Absolute Fairness Gap	$\Theta(s^k)$	$\Theta(s^k)$	$\Theta(s^{-k})$	$\Theta(s^{-k})$
Relative Fairness Gap	$\Theta(1)$	$\Omega(s^k)^\star$	$O(s^{-k})^\star$	$\Theta(1)$

$\star$ If a perfectly fair allocation exists and a technical condition holds

Table 1. The table summarizes the effect on different metrics of an increase in resources for our four general families of group utility functions.

we perform a similar analysis for need-based utilities (linear so homogeneous of degree 1), and observe again that such technical choices have important consequences.

While these observations are for specific utility functions, they apply quite generally. Our observations about need-based utilities apply not just to linear but to any homogeneous function with $k > 0$, so that $u(s\mathbf{r}) = s^k u(\mathbf{r})$, and their corresponding cost minimization version defined in terms of them as $c(s\mathbf{r}) = C - s^k u(\mathbf{r})$. Similarly, our observations about outcome based costs apply not just to the reciprocal but to any homogeneous function of negative degree (which we write as $-k$ for $k > 0$ to make the difference clearer). That is, a cost of the form $c(s\mathbf{r}) = s^{-k} c(\mathbf{r})$ as well as the utility maximization version that results from subtracting the cost from a normalizing constant $u(s\mathbf{r}) = C - s^{-k} c(\mathbf{r})$.

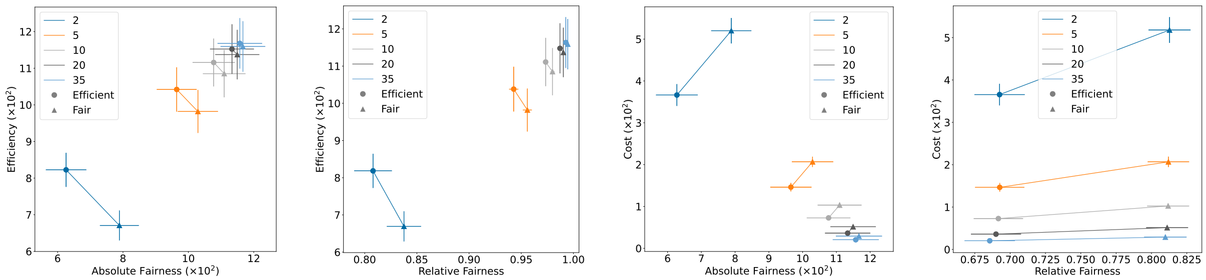
To make this precise, let $\mathbf{r}^*$ be the efficient allocation and $\mathbf{r}$ be the fairest allocation at the same level of resources. Then the efficiency gap, absolute fairness gap, and relative fairness gap are defined as $E(\mathbf{r}^*) - E(\mathbf{r})$, $F_{\text{abs}}(\mathbf{r}, E(\mathbf{r})) - F_{\text{abs}}(\mathbf{r}^*, E(\mathbf{r}^*))$, and $F_{\text{rel}}(\mathbf{r}, E(\mathbf{r})) - F_{\text{rel}}(\mathbf{r}^*, E(\mathbf{r}^*))$ respectively. These capture how far apart the two end points of the Pareto frontier are from each other in the efficiency and fairness dimensions respectively. As in our plots, we can ask how these gaps change with the level of resources. The following theorem gives a characterization that shows the generality of our observations from our stylized examples.

Theorem 1. *Let the optimization goal and form of group utility or cost with degree of homogeneity $k > 0$ be as given in one of the columns of Table 1. Then the difference between the efficient and fair solutions for the metrics listed in the table grow (for need-based) or shrink (for outcome-based) as given in the scale of resources s and degree k.*

In two entries of the table, the relative fairness gap is not homogeneous (although it has an important homogeneous substructure), so a more delicate analysis is needed. For clarity, we break this part of the proof of Theorem 1 into a separate argument. The statement makes precise the technical condition mentioned in Table 1.

Theorem 2. *If $C > u_i(s\mathbf{r})$ (i.e. C is an upper bound on u), the fair solution is perfectly fair but the efficient solution is not, and no group has a cost or utility exactly equal to the average in the efficient solution, then with need-based costs or outcome-based utility the relative fairness gap grows as $\Omega(s^k)$ and shrinks as $O(s^{-k})$ respectively.*

To connect these results to our stylized examples, Figure 3 is an example of the outcome-based case with $k = 1$. In all four subfigures, the vertical gaps between the paired orange and blue point (and for similar reasons the gaps between successive points of the same color) scale as $1/s$. In Subfigures (a) and (c) the horizontal distance between paired orange and blue points (the absolute fairness gap) also scales as $1/s$. Subfigure (d) is the case where relative fairness is constant, while in Subfigure (b) the gap shrinks initially rapidly but eventually more slowly than in (a) which is consistent with the rate being an upper bound. Figure 2, along with Figures 5, 6, and 7 in Appendix D, have the corresponding relation for the need-based case, again for $k = 1$.



(a) Efficiency vs. Absolute Fairness (higher preferred for both).

(b) Efficiency vs. Relative Fairness (higher preferred for both).

(c) Cost (lower preferred) vs. Absolute Fairness.

(d) Cost (lower preferred) vs. Relative Fairness.

Fig. 4. Efficiency-Fairness tradeoffs correspond to various amounts of resources (shown in the legend) for the Chicago Food inspection dataset. The legend shows markers for efficient and fair allocations, and the line color represents the number of resources. Error bars represent standard error from 16-fold cross validation.

To summarize the key takeaways from these results, a key difference between scenarios where utilities are need-based and outcome-based is that in the former the efficiency and fairness gaps grow with the number of resources while in the latter they shrink. In scenarios where either mathematical model may be reasonable, it is important to think about which stakeholders care more about to understand the fairness effects of adjusting the amount of resources available. Along another decision axis, absolute fairness is in some sense more robust than relative fairness, as it gives the same answer regardless of the choice of a utility or cost representation. The $\Theta(1)$ cases are where relative fairness gives an outlier, and in our view, imprecise answer.[5] Thus, we discourage those combinations of modeling choices. However, relative fairness is not without merit: in some scenarios it may better correspond to ethical judgements and the meaning of, e.g., one group having half the utility of the average is easily interpretable even in scenarios where the units of utility (and so the differences used by absolute fairness) are not readily interpretable.

These results focus on the behavior at extreme solutions: maximum efficiency and fairness. In general, we may be interested in the intermediate points between the endpoints. From a theoretical perspective, it is not clear how to pick one of these out to analyze, a limitation that is not unique to our work (e.g., the Price of Fairness also only examines the extremes). That said, if we do commit to an intermediate outcome we prefer at some level of resources, Proposition 2 can provide insights into what allocation will be chosen as the number of resources vary, as long as there is an intermediate objective this choice optimizes that is homogeneous. Theorem 1 still applies in most cases, as the analysis doesn't require that the alternative actually be the perfectly fair one as long as Proposition 2 applies. The cases covered by Theorem 2 have perfect fairness as an assumption, but only one part of the analysis relies on this, so it may be possible to derive at least some insight into the scaling in a particular scenario by adapting the argument.

7 Empirical Study: Chicago Food Inspections

Now, we explore a real-world scenario where noise and other factors imply that utility and objective functions do not precisely satisfy the definition of homogeneity. More broadly, this setting is much richer than our stylized examples (e.g. it has nine groups). Our findings from the empirical study of the Chicago food inspections dataset

[5]See App. A for a similar discussion about the Price of Fairness.

support our insights about the effect of resources on the efficiency-fairness tradeoff and choice of objectives, even when our assumptions hold imperfectly.

Food inspections are part of the services provided by local governments and involve agents (or sanitarians) visiting food establishments and evaluating the conditions as per food safety code [8]. The agents' objective is to identify and help correct any *critical violations* of the food code, as the critical violations pose the highest risk of producing conditions that might result in a foodborne illness. The primary dataset we explore contains more than 18,000 records of inspections done by the agents from the Chicago Department of Public Health (CDPH) over four years, from September 2011 to October 2014 and is publicly accessible on the Chicago Data Portal[6]. Prior studies have used the dataset to train a machine learning model that prioritizes the inspections with a higher likelihood of resulting in critical violations [23, 32, 33]. The model outputs a score that is used to order inspections from the highest score to the lowest to get a schedule that aims to find the critical violations early.

7.1 Methodology for Empirical Study

To study the change in objectives with the number of agents, we use the dataset to create a simulation of the inspection schedule. We describe how we use previous work as our proxies for the efficient and fair allocations for the empirical study, highlighting the differences in stylized examples and a real-world setting.

Efficient Allocation. To schedule the inspections with high likelihoods of *critical violation*, CDPH, along with the help of internal and external organizations, developed a machine learning (ML) model to predict a score for establishments around the city [32]. The food inspections dataset contained additional information like the weather on the inspection day, reports of crime and sanitation in the neighboring area. The department anonymized the identity of the agent who conducted each inspection in the dataset by clustering the agents into groups. The groups were created based on the agent *critical violation rate. Critical violation rate* can be computed as the ratio of the number of inspections with critical violations to the number of inspections completed. This signals how strict is an agent group. There were six of these groups, named after Chicago Transit Authority train lines – Purple, Blue, Orange, Green, Yellow, and Brown. Using all such features, the ML models output a likelihood score for the inspection. The score is then used to order the inspections given a time period, which creates the inspection *schedule*. To evaluate the model, CDPH had a two-month test period where agents did the inspections as usual, and then the model predicted scores for these inspections without the ground truth. Compared to the random schedule by agents, the schedule obtained by ordering the inspections based on the model predictions reduced the average time to detect *critical violations* by 7 days in the simulation study. We use this model as the *efficient allocation* in our setting. More details about the dataset and the experimental setup are described in the work of Schenk Jr. [32][7].

Fair Allocation. Singh et al. [33] studied the impact of the efficient allocation with geographic fairness across nine regions of the city in mind. They found that the proposed ML model learns the signal from the agent group as a proxy for the likelihood of finding a *critical violation* and orders the inspections conducted by the most strict group first while delaying the others. Since there is a correlation between the agent groups and geographic regions, the benefit of scheduling some inspections early at the cost of others puts some regions at a disadvantage. They propose several methods to create a fair schedule by treating the agent groups as the *protected group* for fairness-aware ML models. They also propose two methods that modify the way the predicted score is used to create the schedule while achieving Pareto-dominance on the efficiency-fairness tradeoff. We adapt their 'Sanitarian-blind' approach as our *fair allocation*. The approaches are described in detail in Singh et al. [33][8].

[6]Updated dataset: https://data.cityofchicago.org/Health-Human-Services/Food-Inspections/4ijn-s7e5/about_data. Fixed dataset used for the study: https://github.com/Chicago/food-inspections-evaluation

[7]GitHub: https://github.com/Chicago/food-inspections-evaluation

[8]GitHub: https://github.com/shubhams/eaamo-fair-food

Experimental Setup. The experiments run on a Linux server running Ubuntu 22.04.5 LTS and powered by an Intel Xeon E5-2630 CPU with 32 GB of memory. The code is written in Python 3.12.8 and is available at https://github.com/shubhams/facct26-fair-enough.

Given the original dataset, we want to explore the effect of change in the number of agents on the efficiency and fairness of the proposed solutions. Singh et al. describe the inspections as being conducted by "around three dozen" agents. Using this, we calculate the number of inspections completed per agent per day based on the assumption of 35 agents. Then, for any number of agents, we can compute the number of days required to complete a schedule. We vary the number of agents between $\{n \in \mathbb{Z} \mid 2 \leq n \leq 35\}$ to get a wide range for observing the effects on objectives. We assume that each agent roughly conducts an equal number of inspections, and we use a uniform distribution to assign inspections to agents. We further assume that the number of daily inspections stays uniform. Under this assumption, we use stochastic rounding to split inspections into days. (We show additional sensitivity checks in contrast with the uniform daily inspections assumption with other non-uniform distributions in Appendix F). We use this process for all numbers of agents and simulate an inspection schedule. Since the dataset contains a fixed number of inspections to be completed, the change in the number of agents changes the schedule length − fewer agents would take longer to complete the same amount of inspections and vice versa. For evaluation, we use 16 two-month test windows (following Singh et al. [33]) for cross-validation.

We use outcome-based group utilities and costs. For costs, we follow Schenk Jr. and define the cost of the group as the average number of days that elapse before restaurants with critical violations in the group's region of the city are inspected. Thus, if all were inspected on the first day that cost would be zero while if all were inspected on the last day of the second month it would be approximately sixty. With fewer than 35 agents it would take more than two months to complete all inspections, making the maximum possible cost correspondingly larger. For utility, we instead compute the number of days in the inspection window after a restaurant with a critical violation is inspected. This is equivalent to subtracting the cost from the maximum window length for any amount of agents. Thus, if all were inspected on the first day, that utility would be approximately the time it takes 2 agents to complete the schedule (∼1200 days), while if all were inspected on the last day with two agents it would be zero.

To verify the form of the group utility function, we perform an analysis in Appendix E showing that they follow the outcome-based utility. This allows us to create an analogue of Fig. 3 derived from a real application, shown in Fig. 4.

7.2 Results

Comparing Fig. 3a to Fig. 4a, we see they are consistent in that the increase in resources makes the tradeoff between efficiency and fairness smaller. One key difference from our stylized example in Fig. 3a is that the fair allocation does not achieve the highest possible absolute fairness. This can be explained by the two-step scheduling process: first the ML model is used to get the likelihood score, and then the score is used to order the inspections. However, the model is not aware of the true utility of the scheduling steps and can not be optimized to achieve equal group utilities *ex-post*. Nevertheless, the figure still presents a strong argument for the addition of resources as a way to jointly improve fairness as the tradeoff between the two becomes smaller and the number of agents grows.

Next, we observe the change in tradeoffs when measuring the *relative* fairness instead of absolute fairness, shown in Fig. 4b. The plot reflects our observations from Fig. 3b, the efficient allocation gets closer to the fair allocation as the resources increase with a seemingly smaller benefit of choosing the fair solution under relative fairness than under absolute fairness. The caveat about the fair allocation being unable to achieve perfect fairness still shows up here. In contrast to absolute fairness, relative fairness might lead to the conclusion that even with a low number of resources, the system is already quite fair (note the different scales on the X-axes), and in general, choosing efficient rather than fair allocations appears much more appealing.

We illustrate the effect of resources on a minimizing societal cost function along with both – absolute and relative fairness in Figs. 4c and 4d, respectively. Like the example in Fig. 3c, Fig. 4c shows the tradeoff to be shrinking as the resources are added for *efficient* and *fair* allocations. We note that the fair allocation does not achieve perfect fairness in this case as it is presented with the same challenge of finding out the true utilities *a priori* as when using the maximizing efficiency case (Figs. 4a & 4b). Fig. 4d echoes the findings and limitations of using cost and relative fairness together from Fig. 3d. As the resources increase, the cost of both the efficient and fair allocations decreases. But similar to Fig. 3d, and despite the added complexity of the real scenario, the Fig. 4d tradeoff largely does not change, as predicted by homogeneity. Once again, we could come to the wrong conclusions if we were to choose this combination of efficiency and fairness measures. In this application, both cost and relative fairness individually pose plausible choices, but their combination could lead to adverse decisions.

8 Conclusion

We have argued for considering the effect of resources on the efficiency-fairness tradeoff and provided several insights about the way modeling choices can influence decisions. In particular, we examined how the choices of defining utility functions in terms of need or outcome, defining fairness in absolute or relative terms, and representing goals as benefits or costs. We have seen in both theory and a scenario derived from a real dataset that these technical modeling choices can lead to very different conclusions about the tradeoff between efficiency and fairness and the way it changes with the addition or removal of resources. Furthermore, these differences are predictable and can be understood through the lens of homogeneous functions. We do not provide reductive advice that there is a single "correct" choice: these decisions depend on details of the application domain and the values of those affected. But our work highlights the importance of thinking about them and provides a framework for understanding their consequences, particularly in the context of important societal decisions about resource allocation.

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Endmatter

Generative AI Usage Statement

We, the authors, acknowledge that no Generative AI has been used in the writing or editing any part of this paper.

Ethical Considerations

The objective of this paper is to explain the effect of resources on groups and their utilities and how using them for evaluating the efficiency and fairness of a resource allocation mechanism in a socio-technical system could lead to uninformative conclusions. Such conclusions can misinform decision-making for societal resources that prove valuable to underprivileged groups. Our contributions are majorly driven by understanding the properties of a particular class of functions. We mainly use simple examples to inform the reader about the effect of scaling resources on key objectives – efficiency and fairness. Our experiment section uses a publicly available dataset (described in §7.1) and all identities of the agents were anonymized.

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A Effect of Resources on Price of Fairness

Quantifying the tradeoff between efficiency and fairness is a well-studied problem. We have shown the tradeoff as the Pareto fronts over maximizing and minimizing efficiency functions and *absolute* and *relative* fairness functions. Another popular albeit concise representation of this tradeoff is the *Price of Fairness (PoF)* [4, 10]. PoF is defined[9] as the ratio of the maximum utility from the sets of all solutions S to the maximum utility from the set of fair solutions $F \subseteq S$, given as:

$$\text{PoF} = \frac{\max_{s \in S} \text{efficiency(s)}}{\max_{s \in F} \text{efficiency(s)}} \tag{7}$$

Note this definition computes the ratio of 'efficiency' of the most efficient solution and the fairest solution to specify the tradeoff. Applied to the example we present in Fig. 1, the definition could be rewritten as:

$$\text{PoF} = \frac{\max(E(r))}{\max_{r \in \arg\max F(r)} E(r)} \tag{8}$$

By looking at Fig. 1, we can say that the choice of picking a certain amount of efficiency to trade for fairness would have vastly different consequences for the stakeholders in the three scenarios. However, if we just considered the PoF for the curves, it is uninformative since the ratio of the highest point to the lowest point on the Y-axis (efficiency) does not change as $E(r)$ is scaled. Therefore, if the choice were to be made solely by looking at the PoF for the three curves, the decision maker would not see the difference between the three tradeoff curves.[10]

The invariance of PoF for any change in resources can easily be explained by considering the homogeneity of the function. As stated in Corollary 2, ratios are homogeneous functions of degree 0 and, therefore, do not capture the effect of scaling by a constant factor. In the context of our more general theoretical results, the PoF can be thought of as a ratio-based version of the efficiency gap. Being ratio-based, it has the same issues as the relative fairness gap and we would discourage using it in the same scenarios. Thus, this example presents a setting where, while the PoF provides useful information about the implications of a commitment to fairness, it alone does not portray the complete picture necessary for comparing efficiency-fairness-resources tradeoffs.

B Homogeneity of Common Utility & Fairness Functions

In order for our framework to fully apply, we require both the group utility functions and the (fairness) objectives used to evaluate them to both be homogeneous. For group utility functions, many common choices from the literature, though certainly not all, are homogeneous. Table 2 summarizes the behavior of some popular choices. We also characterize the homogeneity of common utilities used in related work in §4. In this work, our examples have drawn on the linear and monomial cases to examine resources that benefit entire groups.

When considering objectives, the fairness literature has produced a variety of definitions, with the most heavily used including demographic parity and equal opportunity, but there are many others including predictive parity and equalized odds. Originally, this literature focused on classification tasks, which is how many of these concepts are defined. However, it has since been more widely applied. Blandin and Kash [5] give a framework for understanding these generalizations in terms of utility which naturally fits with our approach. By way of example, the original definition of demographic parity requires that the probability of individuals from different groups to receive a positive label is equal. They argue for more generally interpreting it as some measure of the utilities of individuals in the groups being equal. In our context we interpret this a measure as the group utilities

[9] The two papers actually use slightly different definitions, though similar in spirit. We use the latter.

[10] Donahue and Kleinberg [10] more generally allow a bounded amount of unfairness, but similar examples can be constructed. The key point remains that while the PoF can give us some information about the Pareto frontier, it does not tell us how to choose a point on it.

themselves, so we interpret demographic parity as requiring equal group utilities. The definitions of absolute and relative fairness we use are then two natural ways of quantifying the extent to which this equality is violated.

This approach applies beyond demographic parity. Equal opportunity requires the probability of receiving a positive label conditional on the true label being positive to be equal across groups. Blandin and Kash [5] interpret equal opportunity as requiring a measure of utilities conditional on there being a way for the individual to receive a "good" utility being equal. Gölz et al. [20] implement this idea by defining their group utility function in terms of the total number of individuals who would benefit from a resource receiving one. Thus our notions of absolute and relative fairness again measure violations of this desired equality. More broadly, Blandin and Kash [5] argue that a wide range of group fairness notions can be interpreted in terms of equality of appropriately defined measures of utility, and by adopting their approach through appropriate definition of the group utility function our notions of absolute and relative fairness can be applied to a similarly wide range of group fairness concepts.

Mashiat et al. [27] introduce four fairness notions: improvement fairness, gain fairness, regret fairness and shortfall. All enforce an equality between groups and they measure violation of them using absolute fairness (although they use the 1-norm instead of the 2-norm we choose). The difference among them stems from different ways of defining group utilities from individual utilities, and we have already discussed the homogeneity implications of this approach. Specifically, improvement fairness looks at the difference from the individual's worst outcome, gain fairness uses the ratio with the worst outcome, regret fairness defines a cost (the difference from the individual's best possible outcome), and shortfall uses the ratio with the best outcome.

In addition to these objectives inspired by group fairness, there are many ways of defining the social welfare of our collection of groups. We made use of efficiency, also known as utilitarian welfare. But there are other choices such as egalitarian or Nash welfare. There are also inequality measures such as the Gini and Atkinson indicies. All of these are homogeneous. There are also fairness desiderata from the fair division literature such as proportionality, envy-freeness, and equitability that are preserved under scaling of the utilities, even if they are not homogeneous per se because they are not functions. Table 3 summarizes this discussion.

C Proofs

C.1 Proof of Proposition 1

Proof. The statement is direct from the definition of homogeneity. Consider G groups whose utility functions are denoted as $u_g(\cdot)$, and let $\mathbf{r}_g = [r_1, \ldots, r_T]$ denote the resource allocation.

$$f(u_1(s\mathbf{r}_1), \ldots, u_G(s\mathbf{r}_G)) = f(s^{k_1}u_1(\mathbf{r}_1), \ldots, s^{k_1}u_G(\mathbf{r}_G)) = s^{k_1 k_2} f(u_1(\mathbf{r}_1), \ldots, u_G(\mathbf{r}_G)) \qquad \square$$

C.2 Proof of Theorem 1

Proof. We break down the proof into multiple cases based on the type of group utility functions and objectives. Wherever possible, we use symmetry to show the equivalence of cases and avoid repetition. In all cases, by Proposition 2, as the total resources available scale the optimal allocations for both efficiency and fairness scale linearly, making the forms given in the table applicable.

Case 1: Need-based utilities with maximizing goal. For need-based utilities, the efficiency gap is defined as:

$$E(s\mathbf{r}^*) - E(s\mathbf{r}) = \frac{1}{G}\sum_{i=1}^{G} u_i(s{\mathbf{r}^*}_i) - \frac{1}{G}\sum_{i=1}^{G} u_i(s\mathbf{r}_i) = s^k\left(\frac{1}{G}\sum_{i=1}^{G} u_i({\mathbf{r}^*}_i) - \frac{1}{G}\sum_{i=1}^{G} u_i(\mathbf{r}_i)\right)$$

Thus it scales as $\Theta(s^k)$. The absolute fairness gap using need-based utilities is given as:

$$F_{\text{abs}}(\mathbf{sr}, E(\mathbf{sr})) - F_{\text{abs}}(\mathbf{sr}^*, E(\mathbf{sr}^*))$$
$$= \not{C} - \sqrt{\sum_{i=1}^{G}(u_i(s\mathbf{r}_i) - E(\mathbf{sr}))^2} - \left[\not{C} - \sqrt{\sum_{i=1}^{G}(u_i(s\mathbf{r}^*{}_i) - E(\mathbf{sr}^*))^2}\right]$$

By the homogeneity of $E(\cdot)$ (via Corr. 1) and $u(\cdot)$,

$$= s^k\left[\sqrt{\sum_{i=1}^{G}(u_i(\mathbf{r}^*{}_i) - E(\mathbf{r}^*))^2} - \sqrt{\sum_{i=1}^{G}(u_i(\mathbf{r}_i) - E(\mathbf{r}))^2}\right]$$
$$= \Theta(s^k)\ .$$

Finally, we look at the relative fairness gap:

$$F_{\text{rel}}(\mathbf{sr}, E(\mathbf{sr})) - F_{\text{rel}}(\mathbf{sr}^*, E(\mathbf{sr}^*)) = \prod_{i=1}^{G}\left(\frac{\min\left(u_i(s\mathbf{r}_i), E(s\mathbf{r})\right)}{\max\left(u_i(s\mathbf{r}_i), E(s\mathbf{r})\right)}\right)^{\frac{1}{G}} - \prod_{i=1}^{G}\left(\frac{\min\left(u_i(s\mathbf{r}^*{}_i), E(s\mathbf{r}^*)\right)}{\max\left(u_i(s\mathbf{r}^*{}_i), E(s\mathbf{r}^*)\right)}\right)^{\frac{1}{G}}$$
$$= \Theta(1)\ .$$

The final step follows because min and max are homogeneous of degree 1, so by Proposition 1 and Corollary 2 the ratio is homogeneous of degree 0.

This case is symmetric to the case of outcome-based costs, but with the degree of $-k$. Therefore, the proofs for the efficiency, absolute fairness and relative fairness gaps in that case can be derived similarly.

Case 2: Need-based costs. The efficiency gap for the minimization goal is defined as:

$$E(\mathbf{sr}^*) - E(\mathbf{sr}) = \frac{1}{G}\sum_{i=1}^{G}c_i(\mathbf{sr}^*) - \frac{1}{G}\sum_{i=1}^{G}c_i(\mathbf{sr})$$
$$= \frac{1}{G}\sum_{i=1}^{G}\not{C} - u_i(s\mathbf{r}^*{}_i) - \frac{1}{G}\sum_{i=1}^{G}\not{C} - u_i(s\mathbf{r}_i)$$
$$= \frac{s^k}{G}\sum_{i=1}^{G}u_i(s\mathbf{r}_i) - \frac{s^k}{G}\sum_{i=1}^{G}u_i(s\mathbf{r}^*{}_i) = \Theta(s^k)\ .$$

For the absolute fairness gap, we have:

$$F_{\text{abs}}(\mathbf{sr}, E(\mathbf{sr})) - F_{\text{abs}}(\mathbf{sr}^*, E(\mathbf{sr}^*))$$
$$= \not{C} - \sqrt{\sum_{i=1}^{G}(c_i(s\mathbf{r}_i) - E(\mathbf{sr}))^2} - \left[\not{C} - \sqrt{\sum_{i=1}^{G}(c_i(s\mathbf{r}^*{}_i) - E(\mathbf{sr}^*))^2}\right]$$
$$= \sqrt{\sum_{i=1}^{G}\left(\not{C} - s^k u_i(\mathbf{r}^*{}_i) - \left(\not{C} + \frac{1}{G}\sum_{i=1}^{G} -s^k u_i(\mathbf{r}^*)\right)\right)^2} - \sqrt{\sum_{i=1}^{G}\left(\not{C} - s^k u_i(\mathbf{r}_i) - \left(\not{C} + \frac{1}{G}\sum_{i=1}^{G} -s^k u_i(\mathbf{r})\right)\right)^2}$$
$$= \Theta(s^k)\ ,$$

The efficiency and absolute fairness gaps for the outcome-based utility (first and second rows in the third column of Table 1) can also be proved using the same steps as above, and the gap for outcome-based utility decreases as $\Theta(s^{-k})$.

The proof for the starred entries is somewhat different, so we present it separately as Theorem 2.

□

C.3 Proof of Theorem 2

Proof. First we present the argument for the need-based costs case. Consider the inner ratio from Eq. (5) for some group i with resources $s\mathbf{r}$. Define u_+ and u_- as satisfying $\min(c_i(s\mathbf{r}_i), E(s\mathbf{r})) = C - u_+(s\mathbf{r})$ and $\max(c_i(s\mathbf{r}_i), E(s\mathbf{r})) = C - u_-(s\mathbf{r})$. That is, u_- and u_+ are the smaller and larger homogeneous functions of degree k respectively. Then by definition the ratio is of the form

$$\frac{C - u_+(s\mathbf{r})}{C - u_-(s\mathbf{r})} = \frac{C - s^k u_+(\mathbf{r})}{C - s^k u_-(\mathbf{r})} = \frac{u_+(\mathbf{r})}{u_-(\mathbf{r})} - \frac{\frac{u_+(\mathbf{r})}{u_-(\mathbf{r})} - 1}{1 - \frac{u_-(\mathbf{r})}{C} s^k}.$$

Because s only appears once, the dependence on it is essentially of the form $1/(1-x)$. Since $1/(1-x) = 1 + x + x^2 + \ldots$ for $0 \le x < 1$, the overall ratio is decreasing as $\Omega(s^k)$ as long as $\frac{u_-(\mathbf{r})}{C} s^k < 1$ (which holds by assumption) and the numerator of the second term is non-zero (which holds whenever group i does not have exactly the average utility, so again by assumption).

Since each ratio individually is decreasing at $\Omega(s^k)$ in the efficient solution, their geometric mean is also decreasing at $\Omega(s^k)$. When the fair solution is perfectly fair, Eq. (5) becomes 1. In this case the relative fairness gap is therefore increasing at a rate of $\Omega(s^k)$.

For the outcome-based utility case, the same argument shows that the gap is *decreasing* at a rate $O(s^{-k})$ as the lower bound on the rate of increase becomes an upper bound on the rate of decrease. □

D Tradeoff Measures for Linear Utilities

In §6, we explore how the expression of efficiency and fairness with *outcome*-based (inverse) group utilities changes the efficiency-fairness-tradeoff and, consequently, the inferences from the allocation methods. In this section, we present a similar analysis but with *need*-based (linear) utilities.

Fig. 5 presents the tradeoff for efficiency, fairness, and resources when using efficiency and relative fairness. Although the efficiency of both allocation methods increases with the increase in resources, it does not affect the fairness of efficient and fair allocation. Unlike Fig. 3d, this is an accurate reflection of the situation: no matter how many resources we have, giving them all to one group (the efficient solution) is perfectly unfair. This example is somewhat degenerate in this regard.

Now, to transform the linear utilities given in Eq. (1) into a minimizing cost function, we subtract the group utilities from a constant large enough to keep the utilities non-negative, and group 1 remains the privileged group. We write them as:

$$\mathbf{c}_1 = \beta - 2\mathbf{r}_1 , \quad \mathbf{c}_2 = \beta - \mathbf{r}_2 . \tag{9}$$

Using these utilities, we plot the Pareto front for cost and absolute fairness in Fig. 6. We notice a steady decline in the fairness measure, along with the reduction in cost for the efficient allocation with the addition of more resources. The fair allocation retains the highest absolute fairness but also a higher cost when compared to the efficient allocation at the same level of resources.

Finally, we show the tradeoff curve for cost vs. relative fairness in Fig. 7. Similar to Fig. 6, the ends of the Pareto front widen as the efficient allocation gets less fair and cost-effective, while the fair allocation stays perfectly fair and with a lower reduction in cost as the resources are added. However, relative fairness portrays that efficient allocation continues to have a higher level of fairness for fewer resources than when measuring absolute fairness in Fig. 6. Since linear costs are not homogeneous due to the constant β, Fig. 7 does not exhibit the problematic behavior we saw in Fig. 3d.

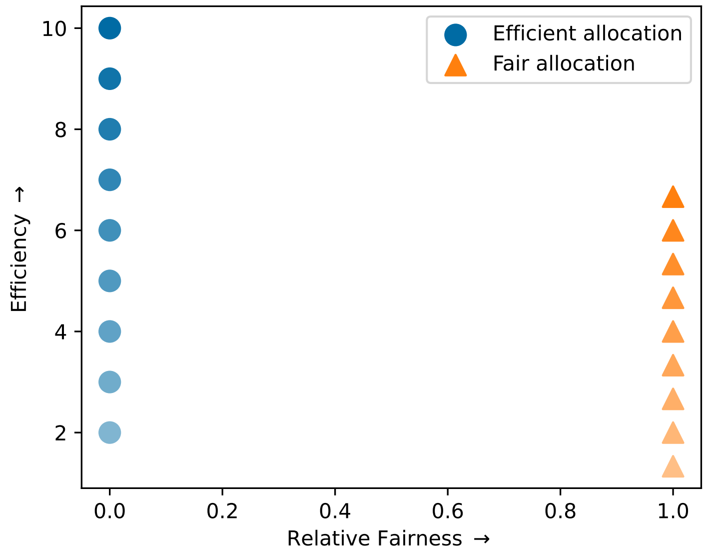


Fig. 5. The three-way tradeoff curves for *need*-based (linear) group utilities measuring efficiency (Y-axis) and relative fairness (X-axis). Higher values for both are better.

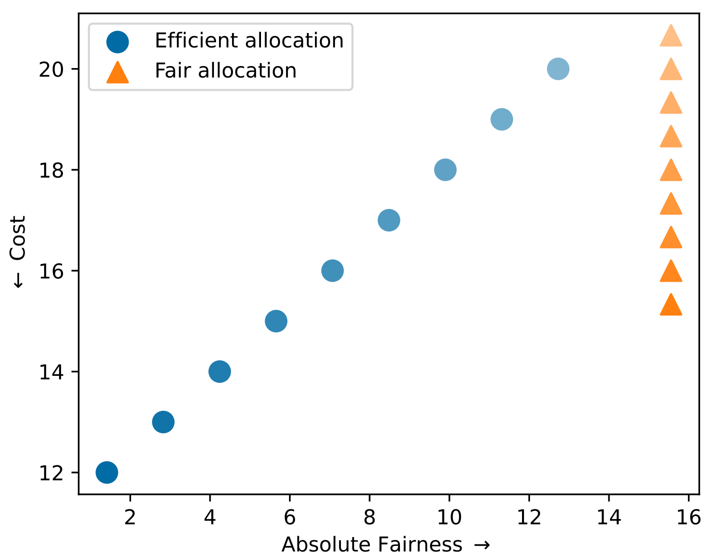


Fig. 6. The tradeoff curves for linear group utilities considering cost (Y-axis) and absolute fairness (X-axis). Lower cost and higher absolute fairness values are better.

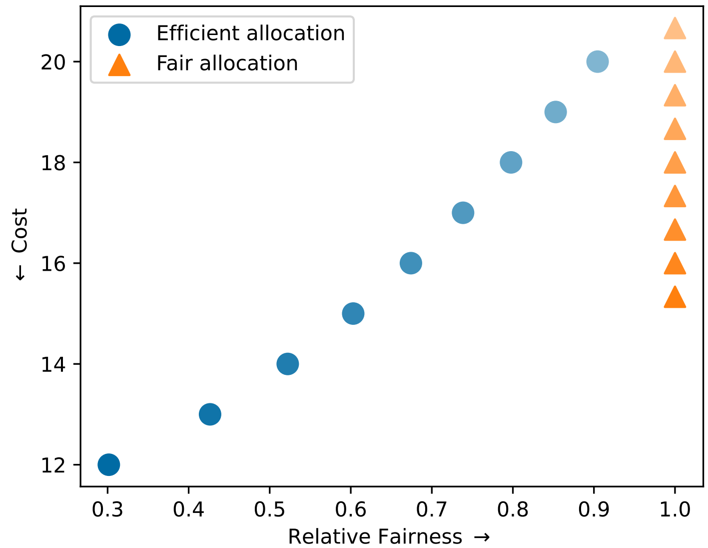


Fig. 7. The tradeoff curves for linear group utilities considering the cost (Y-axis) and relative fairness (X-axis). Lower cost and higher relative fairness values are desired.

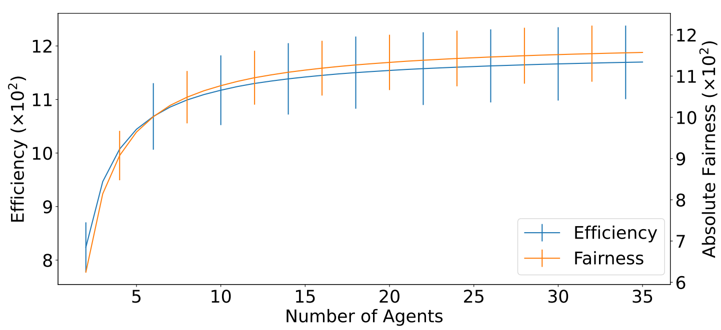


Fig. 8. The plots show the efficiency (left Y-axis) and the absolute fairness (right Y-axis) using the efficient allocation from Schenk Jr. [32]. Both measures increase with the increase in agents (resources). Error bars represent the standard errors.

E Verifying the Objectives for Empirical Study

Keeping the allocation policy fixed to the *efficient allocation* [32], we plot the efficiency and absolute fairness in Fig. 8. The figure shows the effect of change in the number of agents (resources) on the efficiency and absolute fairness separately. At lower levels of resources (between 2-10 agents), both efficiency and fairness show a rapid increase, and slow down approaching the high resources levels (30-35 agents). This shows that the group utilities resemble the outcome-based utilities (Eq. (4)). We show the same allocation by using relative fairness in Fig. 9. Since relative fairness is measured on $[0, 1]$ interval, it appears that efficient allocation achieves higher fairness at lower resource levels and stays comparatively the same as the resources increase. This confirms the objective

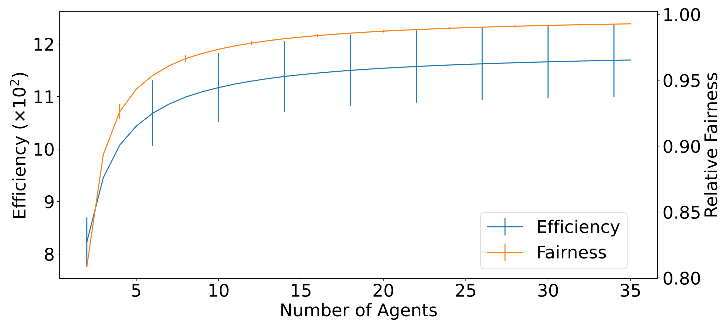


Fig. 9. The plot showing efficiency and fairness, similar to Fig. 8 but uses relative fairness on the right Y-axis. Higher values on both Y-axes are better. Error bars represent standard error.

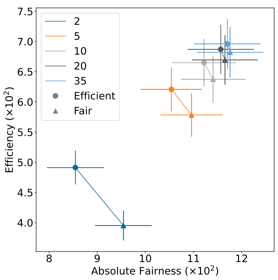
functions we are using in this setting are good approximations of the efficiency (Eq. (2)) and fairness functions (absolute and relative, Eqs. (3), (5)) used in the stylized examples.

F Sensitivity Checks for Food Inspections Simulation

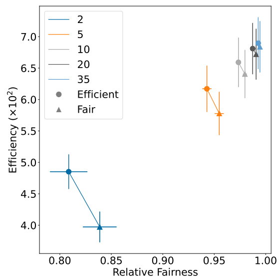
Our empirical results in §7 assume that the number of daily inspections are distributed and scale uniformly as we increase the number of agents. Here, we inspect the effect of non-uniform scaling of the inspections per day. We fit some common distributions (Gamma, log-normal, exponential) with positive support to the inspections dataset and found Gamma to be the best fit based on the sum of squared errors. In addition, we also drew from a Poisson distribution as it is a common method to model the occurrences of independent evens in a given time interval.

Based on the number of inspections per day, we use `stats.gamma.fit()` function in `scipy` [34] to estimate the parameters of a Gamma distribution. Using the maximum likelihood estimate, the best parameter estimates are $\Gamma(\alpha = 3.75, \theta = 7.31)$. To change the number of inspections with the addition of resources, we scale the θ parameter to sample the number of inspections on a specific day in a schedule. The sample is stochastically rounded to make the number of inspections integral. Similarly, we also use the average number of inspections per day in the original food inspections dataset as the λ value to simulate a Poisson distributed daily inspections. For a given amount of resource, we sample daily inspections from $\text{Pois}(\lambda = 27.42)$, where λ gives the expected number of inspections per day.

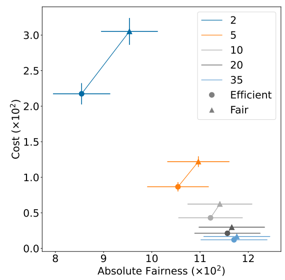
The tradeoff plots are shown in Fig. 10 with Gamma plots in Figs. 10a-10d and Poisson plots in Figs. 10e-10h. We observe that for both the distributions, efficiency and costs has a narrower range compared to that shown in Fig. 4, and hence, lower standard error on the Y-axis. Besides that, we do not notice any significant difference in the behavior of efficiency, fairness and their tradeoffs with the increasing resources when the daily inspections are drawn from Gamma and Poisson distribution, in contrast to, uniform distribution in § 7.1.



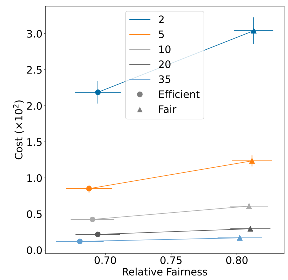
(a) Gamma distribution: Efficiency vs. Absolute Fairness.



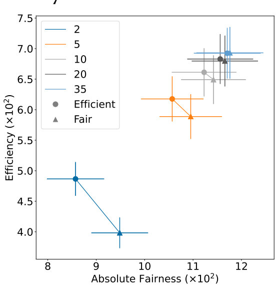
(b) Gamma distribution: Efficiency vs. Relative Fairness.



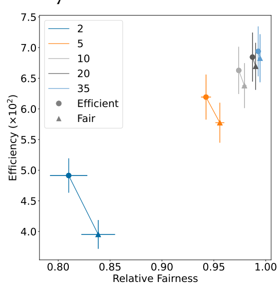
(c) Gamma distribution: Cost vs. Absolute Fairness.



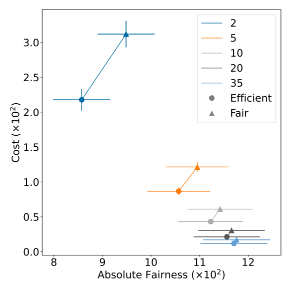
(d) Gamma distribution: Cost vs. Relative Fairness.



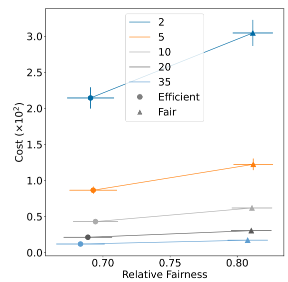
(e) Poisson distribution: Efficiency vs. Absolute Fairness.



(f) Poisson distribution: Efficiency vs. Relative Fairness.



(g) Poisson distribution: Cost vs. Absolute Fairness.



(h) Poisson distribution: Cost vs. Relative Fairness.

Fig. 10. Efficiency-Fairness tradeoffs with daily inspections simulated using Gamma, $\Gamma(\alpha = 3.75, \theta = 7.31)$ (Figs. 10a-10d) and Poisson, $\text{Pois}(\lambda = 27.42)$ (Figs. 10e-10h) distributions with various agents in the Chicago Food inspection dataset.

Table 2. Homogeneity of common utility functions in fair division and resource allocation.

Utility function	Form	Homo-geneous?	Degree	Notes
Linear (additive)	$u(x) = \sum_j v_j x_j$	Yes	ρ	Weights v_j are fixed valuations per unit; scaling the bundle scales utility proportionally.
Monomial (additive)	$u(x) = \sum_j v_j x_j^\rho$	Yes	1	Generalizes linear to allow increasing or diminishing return. Negative ρ capture costs that decrease with resources
Cobb–Douglas	$u(x) = \prod_j x_j^{\alpha_j}, \alpha_j > 0$	Yes	$\sum_j \alpha_j$	Homogeneous of degree $\sum \alpha_j$; degree 1 when weights sum to 1 (constant returns).
Leontief (perfect complements)	$u(x) = \min_j \frac{x_j}{a_j}$	Yes	1	min is positively homogeneous of degree 1; used in fair division with complements.
CES (constant elasticity of substitution)	$u(x) = \left(\sum_j \alpha_j x_j^\rho\right)^{1/\rho}$	Yes	1	Nests linear ($\rho = 1$), Cobb–Douglas ($\rho \to 0$), and Leontief ($\rho \to -\infty$).
Quasi-linear	$u(x, t) = v(x) + t$	Conditional	–	Homogeneous in (x, t) only if v is homogeneous; fails when v is nonlinear (e.g. logarithmic). Common in mechanism design with money.
Additive with diminishing returns	$u(x) = \sum_j v_j(x_j)$, v_j concave	No	–	Concave v_j break scaling: $v_j(sx_j) \neq s\, v_j(x_j)$ in general, although special cases like monomials satisfy. Used in fair division with satiation.
Additive valuation (combinatorial)	$u(S) = \sum_{j \in S} v_j$ for set S	No	–	Defined over discrete bundles; no notion of continuous scaling of a set. Used in combinatorial fair division.
Submodular valuation	$v(S \cup T) + v(S \cap T) \leq v(S) + v(T)$	No	–	Captures complementarities or diminishing returns over sets; inherently discrete.
Lexicographic	$u(x) > u(y)$ iff first differing component of $x > y$	No	–	Not representable as a real-valued function; homogeneity is undefined, although individual components could be homogeneous.

Table 3. Homogeneity of common fairness and social welfare functions in fair division and resource allocation.

Function / criterion	Form	Homo-geneous?	Degree	Notes
(Weighted) Utilitarian welfare	$W = \sum_i \lambda_i u_i, \lambda_i > 0$	Yes	1	Weights allow priority or normalization. $\lambda_i = 1$ is Efficiency
Nash welfare (Nash product)	$W = \prod_i u_i$	Yes	G	Used in Nash bargaining and MNW allocations.
Geometric mean welfare	$W = (\prod_i u_i)^{1/G}$	Yes	1	Normalized Nash welfare
Egalitarian welfare (max-min)	$W = \min_i u_i$	Yes	1	Used in egalitarian fair division and Rawlsian welfare.
Gini index	$\frac{1}{2G^2 E(\mathbf{r})} \sum_{i,j} \lvert u_i - u_j \rvert$	Yes	0	Common measure of inequality.
Atkinson index	$1 - \frac{1}{E(\mathbf{r})} \left(\frac{1}{G} \sum_i u_i^{1-\varepsilon} \right)^{1/(1-\varepsilon)}$	Yes	0	Common measure of inequality.
Relative Fairness	$\prod_{i=1}^{G} \left(\frac{\min(\mathbf{u}_i, E(\mathbf{r}))}{\max(\mathbf{u}_i, E(\mathbf{r}))} \right)^{\frac{1}{G}}$.	Yes	0	One way to quantify violations of group fairness
Absolute Fairness	$C - \sqrt{\sum_{i=1}^{G} (\mathbf{u}_i - E(\mathbf{r}))^2}$.	Conditional	1	Another way to quantify violations of group fairness. Only homogeneous if $C = 0$. However the difference between absolute fairness of two allocations is always homogeneous. Can also use other choices of norm.
Proportionality	$u_i \geq \frac{1}{G} u_i^*$	Conditional	–	Preserved under scaling; but it is a criterion, not an objective.
Envy-freeness	$u_i(\mathbf{r_i}) \geq u_i(\mathbf{r_j}) \ \forall\, i, j$	Conditional	–	Preserved under scaling; but it is a criterion, not an objective.
Equitability	$u_i = u_j \ \forall\, i, j$	Conditional	–	Preserved under scaling; but it is a criterion, not an objective.

Legend: Homogeneous | Conditionally homogeneous or criterion-only | Not homogeneous.